Cambridge Assessment
Admissions Testing

Test of Mathematics for University Admission

Paper 1 specimen paper worked answers

Test of Mathematics for University Admission, Specimen Paper 1 Worked Solutions

Version 1.1, November 2020
Question 6 amended

Contents

Introduction for students

These solutions are designed to support you as you prepare to take the Test of Mathematics for University Admission. They are intended to help you understand how to answer the questions, and therefore you are strongly encouraged to **attempt the questions first** before looking at these worked solutions. For this reason, each solution starts on a new page, so that you can avoid looking ahead.

The solutions contain much more detail and explanation than you would need to write in the test itself – after all, the test is multiple choice, so no written solutions are needed, and you may be very fluent at some of the steps spelled out here. Nevertheless, doing too much in your head might lead to making unnecessary mistakes, so a healthy balance is a good target!

There may be alternative ways to correctly answer these questions; these are not meant to be 'definitive' solutions.

The questions themselves are available on the 'Preparing for the test' section on the Admissions Testing website.

Question 1

We can rearrange the first equation to

$$x = 3y - 1.$$

Substituting this into the second equation gives

$$3(3y-1)^2 - 7(3y-1)y = 5$$

which expands to

$$27y^2 - 18y + 3 - 21y^2 + 7y = 5$$

so

$$6y^2 - 11y - 2 = 0.$$

We can either factorise this as $(y-2)(6y+1) = 0$, giving $y = 2$ or $y = -\frac{1}{6}$ or use the quadratic formula to obtain

$$y = \frac{11 + \sqrt{11^2 - 4 \times 6 \times (-2)}}{12} = \frac{11 \pm 13}{12}$$

giving $y = 2$ or $y = -\frac{1}{6}$ again.

These give $x = 5$ or $x = -\frac{3}{2}$, so the sum is 3.5, and the answer is D.

Alternatively, we could have rearranged the first equation to get

$$y = \frac{x+1}{3}$$

and then substitute this into the second equation. This has the advantage that we obtain the values of x directly, but the disadvantage that there are fractions involved throughout.

Question 2

We rewrite $\sin^2\theta$ as $1-\cos^2\theta$ to give

$$1-\cos^2\theta+3\cos\theta=3.$$

This is now a quadratic in $u=\cos\theta$, giving

$$1-u^2+3u=3$$

or $u^2-3u+2=0$. This factorises as $(u-1)(u-2)=0$, so $u=1$ or $u=2$, that is, $\cos\theta=1$ or $\cos\theta=2$.

$\cos\theta=2$ has no real solutions. The solutions of $\cos\theta=1$ in the given range are $\theta=0$, $\theta=2\pi$, $\theta=4\pi$. So there are three solutions, and the answer is option D.

Question 3

We find the equation of the perpendicular bisector first, and then find the x-coordinate of this line when $y = 0$.

The midpoint of the line segment joining $(2, -6)$ and $(5, 4)$ is $(\frac{7}{2}, -1)$, and the gradient of this line segment is $\frac{4-(-6)}{5-2} = \frac{10}{3}$.

Therefore the perpendicular bisector passes through $(\frac{7}{2}, -1)$ and has gradient $-\frac{3}{10}$. Its equation is therefore

$$y - (-1) = -\tfrac{3}{10}\left(x - \tfrac{7}{2}\right).$$

We could expand and simplify this equation, but that is not necessary for our purposes. Instead, we substitute $y = 0$ to obtain

$$1 = -\tfrac{3}{10}\left(x - \tfrac{7}{2}\right)$$

giving

$$x - \tfrac{7}{2} = -\tfrac{10}{3}$$

so $x = \frac{7}{2} - \frac{10}{3} = \frac{21}{6} - \frac{20}{6} = \frac{1}{6}$ showing that the correct answer is option B.

Question 4

We can factorise the given inequality to give $(x+1)(x-1)(x-2) > 0$. The function $f(x) = (x+1)(x-1)(x-2)$ is zero when $x = -1$, $x = 1$ and $x = 2$.

We can now make a table showing the signs of the three factors for different values of x. (This is a useful technique in general.)

	$x < -1$	$x = -1$	$-1 < x < 1$	$x = 1$	$1 < x < 2$	$x = 2$	$x > 2$
$x+1$	$-$	0	$+$	$+$	$+$	$+$	$+$
$x-1$	$-$	$-$	$-$	0	$+$	$+$	$+$
$x-2$	$-$	$-$	$-$	$-$	$-$	0	$+$
$f(x)$	$-$	0	$+$	0	$-$	0	$+$

Therefore $f(x)$ is positive when $-1 < x < 1$ and when $x > 2$, so the answer is E.

Question 5

$$
\begin{aligned}
& & y &= -\log_{10}(1-x) & & \\
&\iff & -y &= \log_{10}(1-x) & & \text{negate} \\
&\iff & 10^{-y} &= 1-x & & \text{exponentiate to base 10} \\
&\iff & x + 10^{-y} &= 1 & & \text{add } x \\
&\iff & x &= 1 - 10^{-y} & & \text{subtract } 10^{-y}
\end{aligned}
$$

so the answer is D.

Question 6

Since $x + 2$ is a factor, substituting $x = -2$ into the polynomial must yield zero by the factor theorem:

$$(-2)^3 + 4c(-2)^2 + (-2)(c+1)^2 - 6 = 0.$$

Simplifying gives

$$-8 + 16c - 2(c^2 + 2c + 1) - 6 = 0$$

so

$$-2c^2 + 12c - 16 = 0.$$

Dividing by -2 now gives

$$c^2 - 6c + 8 = 0$$

so $(c-2)(c-4) = 0$ and the roots are $c = 2$ and $c = 4$, with a sum of 6. Hence the answer is D.

We could also find the sum of the roots directly from the quadratic $c^2-6c+8 = 0$ without solving it: if the roots are $c = p$ and $c = q$, then the quadratic can be written as $(c-p)(c-q) = 0$, which expands to $c^2 - (p+q)c + pq = 0$. So the sum of the roots is the negative of the c coefficient, which is 6, and the product of the roots is the constant, which is 7.

Question 7

We could do this using a tree diagram, but it would have 3×3 branches, which is somewhat unwieldly in a short test.

Here are two alternative approaches.

Approach 1: use symmetry

The first ball can be anything; it does not matter what. We are left with $n-1$ balls of the same colour and $2n$ balls of different colours; there are $3n-1$ balls left in total. (This is because there are equal numbers of each colour to begin with. Were there different numbers of different colours, we would have to treat each colour separately.)

So the probability that the two balls are not the same colour is $\dfrac{2n}{3n-1}$, which is option C.

Approach 2: use negation

The probability that the two balls not the same colour is 1 minus the probability that they are the same colour. This gives

$$1 - \left(\frac{n}{3n} \times \frac{n-1}{3n-1} + \frac{n}{3n} \times \frac{n-1}{3n-1} + \frac{n}{3n} \times \frac{n-1}{3n-1}\right)$$

where we have added the three colours separately. This simplifies to

$$1 - \frac{n-1}{3n-1} = \frac{2n}{3n-1}.$$

We could have simplified the calculation in the same way as in approach 1: the probability that the second ball is the same colour as the first is $\dfrac{n-1}{3n-1}$, so the probability that they are different colours is 1 minus this.

Question 8

The options all involve logarithm to base 10, so we start by taking logarithms of the given equation. This gives

$$\log_{10}(a^x b^{2x} c^{3x} = \log_{10} 2$$

so

$$\log_{10}(a^x) + \log_{10}(b^{2x}) + \log_{10}(c^{3x}) = \log_{10} 2.$$

We now use the index rule for logarithms to give

$$x \log_{10} a + 2x \log_{10} b + 3x \log_{10} c = \log_{10} 2$$

which factorises to

$$x(\log_{10} a + 2 \log_{10} b + 3 \log_{10} c) = \log_{10} 2$$

and hence

$$x = \frac{\log_{10} 2}{\log_{10} a + 2 \log_{10} b + 3 \log_{10} c}.$$

This, though, is not one of the offered options; we now have to apply the logarithm rules in reverse to give

$$x = \frac{\log_{10} 2}{\log_{10} a + \log_{10}(b^2) + \log_{10}(c^3)}$$

which simplifies further to

$$x = \frac{\log_{10} 2}{\log_{10}(ab^2c^3)}$$

and so the answer is F.

Question 9

The roots of the equation, using the quadratic formula, are

$$x = \frac{11 \pm \sqrt{11^2 - 8c}}{4}$$

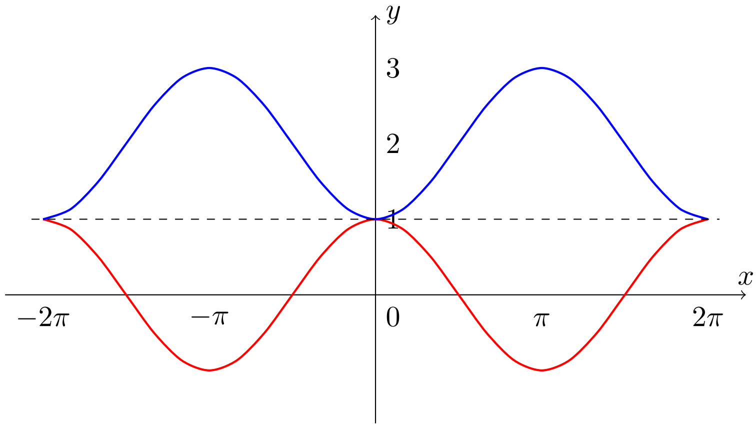
and these differ by

$$x = \frac{2\sqrt{11^2 - 8c}}{4} = 2$$

so we require $\sqrt{11^2 - 8c} = 4$, or $121 - 8c = 16$. Thus $8c = 105$, so $c = \frac{105}{8}$ and the answer is A.

Question 10

When the graph is reflected in the line $y = 1$, we obtain the following:



We can find the equation of the reflected curve in a variety of ways. One is to observe that the reflection is centred about $y = 2$, as the values go from $y = 1$ to $y = 3$. The cosine curve is 'upside-down', so the equation must be $y = -(\cos x) + 2 = 2 - \cos x$.

Another way to see this is as follows. The refletion of $y = \mathrm{f}(x)$ in the x-axis ($y = 0$) is $y = -\mathrm{f}(x)$. When the line of reflection is translated, the whole reflection is translated. Since $y = 0$ is transformed to $y = 2$, the reflection of the function must be translated to give $y = 2 - \mathrm{f}(x)$.

The result is then translated by $\frac{\pi}{4}$ in the positive x-direction, so that x is replaced by $x - \frac{\pi}{4}$. (We can see this because the new $x = \frac{\pi}{4}$ corresponds to the old $x = 0$.)

Therefore, the equation of the resulting graph is

$$y = 2 - \cos\left(x - \frac{\pi}{4}\right).$$

Thus the correct answer is option D.

Question 11

Writing $y = 2^x$, the equation becomes $y^2 - 8y + 15 = 0$, which factorises to give $(y-3)(y-5) = 0$. Thus $y = 3$ or $y = 5$, so $2^x = 3$ or $2^x = 5$.

The given options all use logarithms to base 10, so we will take logarithms of these two equations to base 10. The first gives $x \log_{10} 2 = \log_{10} 3$, so $x = \dfrac{\log_{10} 3}{\log_{10} 2}$. Likewise, the second equation gives $x = \dfrac{\log_{10} 5}{\log_{10} 2}$. Finally, we can add these to obtain

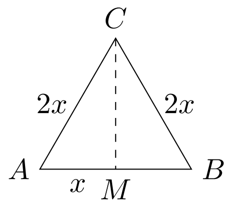
$$\frac{\log_{10} 3}{\log_{10} 2} + \frac{\log_{10} 15}{\log_{10} 2} = \frac{\log_{10}(3 \times 5}{\log_{10} 2} = \frac{\log_{10} 15}{\log_{10} 2}$$

and the answer is option E.

An alternative is to take logarithms to base 2, giving the roots as $\log_2 3$ and $\log_2 5$. These sum to $\log_2 15$. Unfortunately, this is not one of the options offered, as they are all given using logarithms to base 10. If you know the change of base rule for logarithms, though, which states that $\log_a b = \dfrac{\log_c b}{\log_c a}$, you can put $c = 10$ to obtain the correct option.

Question 12

The volume of the prism is the length times the cross-sectional area, so we need to work out the area of the equilateral triangle:



We could use trigonometry and the formula for the area of a triangle, area $= \frac{1}{2}ab\sin C$; this gives the area as $\frac{1}{2}(2x)(2x)\sin 60° = x^2\sqrt{3}$.

Alternatively, we could find the length CM using Pythagoras's theorem, giving $CM^2 = (2x)^2 - x^2 = 3x^2$, so $CM = x\sqrt{3}$. Thus the area of the triangle is $\frac{1}{2}AB \times CM = x^2\sqrt{3}$.

Thus the volume of the prism is $T = x^2 d\sqrt{3}$.

The total surface area of the prism is twice the area of the triangle, plus the area of the three rectangular faces, so

$$T = 2x^2\sqrt{3} + 3(2xd) = 2x^2\sqrt{3} + 6xd.$$

These expressions for T are equal, so

$$x^2 d\sqrt{3} = 2x^2\sqrt{3} + 6xd.$$

Collecting the d terms to the left hand side gives

$$x^2 d\sqrt{3} - 6xd = 2x^2\sqrt{3}$$

so

$$d(x^2\sqrt{3} - 6x) = 2x^2\sqrt{3}$$

hence

$$d = \frac{2x^2\sqrt{3}}{x^2\sqrt{3} - 6x} = \frac{2x\sqrt{3}}{x\sqrt{3} - 6}.$$

If we now divide the numerator and denominator by $\sqrt{3}$, we obtain

$$d = \frac{2x}{x - 2\sqrt{3}},$$

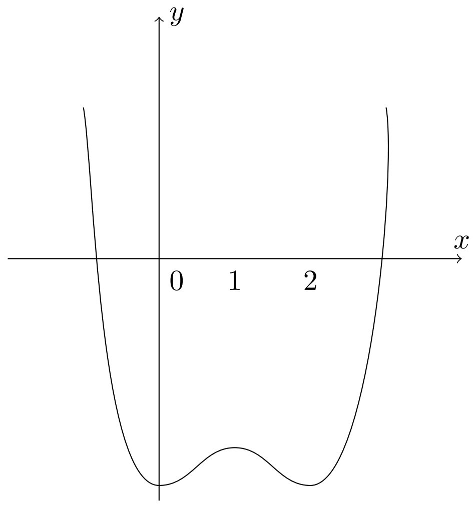
which is option D.

Question 13

We can determine the answer to this question by locating the stationary points of the function $\mathrm{f}(x) = x^4 - 4x^3 + 4x^2 - 10$. We have

$$\mathrm{f}'(x) = 4x^3 - 12x^2 + 8x = 4x(x^2 - 3x + 2) = 4x(x-1)(x-2).$$

There are therefore stationary points at $x = 0$, $x = 1$ and $x = 2$. Substituting these values into $\mathrm{f}(x)$ gives us the y-coordinates of these points on the graph of $y = \mathrm{f}(x)$: they are $(0, -10)$, $(1, -9)$ and $(2, -10)$. Since $\mathrm{f}(x)$ tends to $+\infty$ as x tends to $+\infty$ or $-\infty$, the graph looks roughly like this:



There are thus only two real solutions to $\mathrm{f}(x) = 0$, and the answer is option C.

Question 14

If the graph is a straight line, then we must have

$$(\log y) = m(\log x) + c$$

for some m and c; this is just the usual straight-line equation with x and y replaced by $\log x$ and $\log y$.

This can be written as

$$\log y = \log(x^m) + c.$$

If we now write $c = \log C$ for some C, then this becomes

$$\log y = \log x^m + \log C = \log(Cx^m).$$

Exponentiating both sides gives

$$y = Cx^m$$

which is in the form of option D.

An alternative to writing $c = \log C$ is just to exponentiate the equation $\log y = \log(x^m) + c$. If we suppose that the base of the logarithms is k, then this gives

$$y = x^m k^c$$

(using $k^{u+v} = k^u k^v$ and $k^{\log u} = u$ for any u and v). Since k^c is a constant, this equation can be rewritten as $y = ax^m$ where $a = k^c$, and we again obtain option D.

Question 15

We start by evaluating the integral to obtain

$$\begin{aligned}\int_0^1 (x-a)^2 \, \mathrm{d}x &= \int_0^1 x^2 - 2ax + a^2 \, \mathrm{d}x \\ &= \left[\tfrac{1}{3}x^3 - ax^2 + a^2x\right]_0^1 \\ &= \tfrac{1}{3} - a + a^2\end{aligned}$$

We can now complete the square on this final expression to obtain

$$(a - \tfrac{1}{2})^2 - (\tfrac{1}{2})^2 + \tfrac{1}{3} = (a - \tfrac{1}{2})^2 + \tfrac{1}{12}$$

and so the smallest possible value is $\frac{1}{12}$, which is option A.

We could alternatively have used calculus to find the stationary point of $\frac{1}{3} - a + a^2$ (by differentiating with respect to a), and then noting that this is a local minimum, and that the function is greater than this at all other points.

Question 16

We begin by simplifying this expression; we write each term as a product of prime powers to obtain:

$$\begin{aligned}
\frac{10^{c-2d}\times 20^{2c+d}}{8^c\times 125^{c+d}} &= \frac{(2\times 5)^{c-2d}\times (2^2\times 5)^{2c+d}}{(2^3)^c\times (5^3)^{c+d}} \\
&= \frac{2^{c-2d+2(2c+d)}\times 5^{c-2d+(2c+d)}}{2^{3c}\times 5^{3(c+d)}} \\
&= \frac{2^{5c)}\times 5^{3c-d}}{2^{3c}\times 5^{3c+3d}} \\
&= 2^{2c}\times 5^{-4d}
\end{aligned}$$

For this to be an integer, we require $2c$ and $-4d$ to be non-negative integers. Since c and d are non-zero integers, we need $c > 0$ and $d < 0$, which is option E.

(In fact, this is an "if and only if" condition; options C, D and F would make the expression non-integer, as would A and G; while conditions B and H are necessary, they are not sufficient: if $d < 0$, it is still possible that $c < 0$, so it is not true that the given expression is (necessarily) an integer if $d < 0$.)

Question 17

The quadratic equation can be written as $ax^2 + (a-2)x - 2 = 0$, which has discriminant

$$(a-2)^2 - 4a(-2) = a^2 - 4a + 4 + 8a = a^2 + 4a + 4 = (a+2)^2.$$

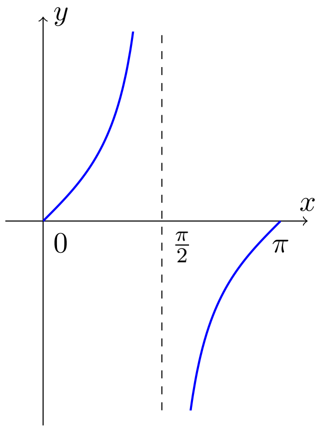
The quadratic has real distinct roots when the disciminant is positive, so when $(a+2)^2 > 0$.

Now $(a+2)^2 = 0$ when $a = -2$, and $(a+2)^2 \geqslant 0$ for all values of a, so $(a+2)^2 > 0$ when $a \neq -2$, which is option D.

Question 18

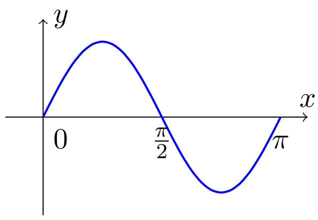
We first consider each of the inequalities separately.

A sketch of the graph of $y = \tan x$ for $0 \leqslant x \leqslant \pi$ is as follows:



Noting that $\tan \frac{pi}{4} = 1$ and $\tan \frac{3\pi}{4} = -1$, we see from the sketch that $-1 \leqslant \tan x \leqslant 1$ when $0 \leqslant x \leqslant \frac{\pi}{4}$ and when $\frac{3\pi}{4} \leqslant x \leqslant \pi$.

Next, we sketch the graph of $y = \sin 2x$ over the same interval:



Noting that $\sin \frac{\pi}{6} = \sin \frac{5\pi}{6} = 0.5$, we see from the sketch that $\sin 2x \geqslant 0.5$ when $\frac{\pi}{6} \leqslant 2x \leqslant \frac{5\pi}{6}$, that is, when $\frac{\pi}{12} \leqslant x \leqslant \frac{5\pi}{12}$.

Now comparing the intervals for the tangent and sine inequalities, we note that they are both satisfied when $\frac{\pi}{12} \leqslant x \leqslant \frac{\pi}{4}$, and the length of this interval is $\frac{\pi}{4} - \frac{\pi}{12} = \frac{\pi}{6}$, so the answer is B.

Question 19

The first four terms are 4, $4r$, $4r^2$ and $4r^3$. We are also given that 4, $4r$ and $4r^3$ are three successive terms of an arithmetic series. We thus have

$$\begin{aligned} 4r &= 4 + d \\ 4r^3 &= 4 + 2d \end{aligned}$$

where d is the common difference of this arithmetic series.

Therefore $d = 4(r - 1)$ from the first equation, and $d = 4r(r^2 - 1)$ on subtracting the two equations.

Equating these expressions for d gives

$$4(r - 1) = 4r(r^2 - 1)$$

so $r - 1 = r(r - 1)(r + 1)$ on factorising and dividing by 4.

Since $0 < r < 1$, we can divide by $r - 1$ to obtain $1 = r(r + 1)$, so $r^2 + r - 1 = 0$. The solutions to this quadratic are

$$r = \frac{-1 \pm \sqrt{1^2 + 4}}{2} = \frac{-1 \pm \sqrt{5}}{2}.$$

Since $r > 0$, we have $r = \frac{-1+\sqrt{5}}{2}$, and so the sum to infinity (given by the standard formula $S_\infty = \frac{a}{1-r}$) is

$$\begin{aligned} \frac{4}{1 - \frac{-1+\sqrt{5}}{2}} &= \frac{8}{2 - (-1 + \sqrt{5})} \\ &= \frac{8}{3 - \sqrt{5}} \\ &= \frac{8(3 + \sqrt{5})}{9 - 5} \\ &= 2(3 + \sqrt{5}) \end{aligned}$$

and the answer is D.

Question 20

We expand each of the terms up to the term in x^2 using the binomial theorem:

$$\begin{aligned}(1+2x+3x^2)^6 &= 1+6(2x+3x^2)+15(2x+3x^2)^2+\cdots \\ &= 1+12x+18x^2+15(4x^2+\cdots)+\cdots \\ &= 1+12x+78x^2+\cdots \\ (1+4x^3)^5 &= 1+5(4x^3)+\cdots = 1+\cdots\end{aligned}$$

so the whole expression is

$$\begin{aligned}(4-x^2)[(1+2x+3x^2)^6-(1+4x^3)^5] &= (4-x^2)[(1+12x+78x^2+\cdots)-(1+\cdots)] \\ &= (4-x^2)(12x+78x^2+\cdots) \\ &= 4(12x+78x^2+\cdots)-\cdots \\ &= 48x+312x^2+\cdots\end{aligned}$$

so the coefficient of x^2 is 312, and the answer is G.

This document was initially designed for print and as such does not reach accessibility standard WCAG 2.1 in a number of ways, including missing text alternatives and missing document structure.

If you need this document in a different format, please email admissionstesting@cambridgeassessment.org.uk telling us your name, email address and requirements and we will respond within 15 working days.

Cambridge Assessment
Admissions Testing
The Triangle Building
Shaftesbury Road
Cambridge
CB2 8EA
United Kingdom

Admissions tests support:

admissionstesting.org/help